The Cosmological Constant

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Abstract

Contrary to popular mythology, Einstein did *not* invent the cosmological constant just in order construct his model universe. He discussed it earlier in "The Foundations of General Relativity"in connection with the proper structure of the source-free field equations. There he dismissed it as arbitrary and unnecessary. It was later that he found its inclusion to be essential to the construction of his model.

Contrary to virtually universal belief, the cosmological constant was *not* invented by Einstein [1] for use in his *cosmological Considerations Concerning General Relativity* [2]. The fact is, he discussed it explicitly about a year earlier in Sect. (14) of *The Foundations of General Relativity* [3] where he introduced the source-free field equations.

$$\frac{\partial \Gamma^{\alpha}{}_{\mu\nu}}{\partial x_{\alpha}} + \Gamma^{\alpha}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\alpha} = 0 \tag{1a}$$

$$\sqrt{-g} = 1\,. \tag{1b}$$

(The second equation is a "coordinate condition"which permits writing the field equations elegantly if severely truncated. In this form, the field equations have never been applied to any problem.) There he comments,

> "It must be pointed out that there is only a minimum of arbitrariness in the choice of these equations. For besides $G_{\mu\nu}$ there is no tensor of second rank which is formed from the $g_{\mu\nu}$ and its derivatives, contains no derivatives higher than the second and is linear [4] in these derivatives."

Then he adds a footnote,

> "Properly speaking this is true only of the tensor
>
> $$G_{\mu\nu} + \lambda g_{\mu\nu} g^{\alpha\beta} G_{\alpha\beta} \tag{2}$$
>
> where λ is a *constant* [emphasis added]. If, however we set this tensor $= 0$, we come back again to the equation $G_{\mu\nu} = 0$."

The term in λ is is not the optimal choice. In modern notation, $G_{\mu\nu}$ is the Ricci tensor so that Exp. (2) would be

$$R_{\mu\nu} + \lambda g_{\mu\nu} R\,. \tag{3}$$

This would result in the source-free equations

$$R_{\mu\nu} = \frac{1}{4} g_{\mu\nu} R\,. \tag{4}$$

The more desirable expression is the simpler $\lambda g_{\mu\nu}$. In fact, the most general tensor satisfying the conditions mentioned earlier and which has a vanishing divergence is [5] is (the now standard)

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu}\,. \tag{5}$$

In the source-free case this reduces readily to

$$R_{\mu\nu} = \frac{\lambda}{4} g_{\mu\nu}\,. \tag{6}$$

These have been studied by Petrov [6] who termed the solutions, "Einstein Spaces".

Einstein sorted this out in the *Cosmological Considerations* .... In Eq. (13) therein he included the cosmological constant

$$R_{\mu\nu} - \lambda g_{\mu\nu} = -\kappa (T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T) \tag{7}$$

which is readily converted to

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} \ = -\kappa T_{\mu\nu}\,.$$

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References

[1] Its derisive appellation, "Einstein's Fudge Factor", if used as a search parameter, will generate over 1.4×10^6 hits.

[2] A. Einstein, “Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie”, *Sitzungsberichte der Preussischen Akademie der Wissenschaften* 170 (1917), Berlin. An English translation appears in the collection of papers, *The Principle of Relativity*, Dover Publications, New York (1923).

[3] A. Einstein, “Die Grundlage der Allgemeine Relativitätstheorie”, *Annalen der Physik*, **49**, 769 (1916). An English translation appears in *The Collected Papers of Albert Einstein*, vol. 6, A. Engel, translator, E.L.Schucking, consultant, 146-200, (1999) Princeton University Press, Princeton, NJ. It also appears in translation in *The Principle of Relativity*, Dover Publications, New York (1923).

[4] This theorem may well have been furnished by Marcel Grossmann whom Einstein credits for help with the mathematics of the paper. The requirement of linearity is a necessary and sufficient condition for the equations to have unique solutions. See, for instance, Adler *et al*, *Introduction to General Relativity*, 2nd ed., Sect. 5.1, McGraw-Hill, New York (1975).

[5] D. Lovelock, “The Four Dimensionality of Space and the Einstein tensor”, *Journal of Mathematical Physics.*, **13**, 874-876 (1972); D. Lovelock and H. Rund, *Tensors, Differential Forms, and Variational Principles*, Dover Publications, NY (1989), pp. 314-323.

[6] A. Z. Petrov, *Einstein Spaces*, Pergamon Press, London (1969).